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Low-frequency expansions for a penetrable ellipsoidal scatterer in an elastic medium*

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Abstract. In this paper the scattering of a plane longitudinal or transverse wave by a penetrable ellipsoid in an isotropic and homogeneous elastic medium in the low-frequency region is examined. Using low-frequency expansions the scattering problem is reduced to a sequence of potential problems. Explicit closed-form solutions for the zeroth and first-order approximations are obtained. The solution of the problem was made possible by using an analytical technique based on Papkovich–Grodski–Neuber potentials. The normalized scattering amplitudes and the scattering cross-section are evaluated up to k^3-order and k^4-order terms, respectively.

1. Introduction

The scattering of a plane harmonic wave in linear elasticity is an exterior boundary-value problem for the Navier equation, with known boundary conditions on the surface of the scatterer and prescribed radiation conditions. The general theory of scattering of elastic waves is very well exposed by Kupradze [1], who gave integral representations and radiation conditions. Integral representations are also given by Pao and Varatharajulu [2]. Uniqueness theorems in elasticity are proved by Wheeler and Sternberg [3]. Barratt and Collins [4] were the first to give relations for the scattering cross-section. For the fundamental scattering theorems we refer to [5].

The scattering of a longitudinal wave by a sphere was investigated for the first time by Ying and Truell [6]. Einspruch, Witterholt and Truell [7] have also solved the corresponding problem for transverse incidence. Lawrence [8] used an analytical technique to evaluate the leading low-frequency term for the scattering cross-section of an ellipsoid. Results for the cases of the rigid scatterer and the cavity for a longitudinal or transverse incident wave have also been presented in [9, 10, 11]. Results about the low-frequency scattering theory for a penetrable body are given in [12]. Estimates for the accuracy of the coefficients in the low-frequency expansion are given by Jones [13].

In this paper, the scattering of a plane harmonic elastic longitudinal or transverse wave by a penetrable scatterer of ellipsoidal shape is examined. The boundary conditions require that on the surface of the scatterer the displacement and the traction fields have to be continuous. Using expansions in the low-frequency region, the wave problem is reduced to a sequence of potential problems, which by means of Papkovich potentials for the displacement fields can be solved recursively. The zeroth as well as the first-order approximation of the solution are obtained in closed form. Many technical difficulties, which reflect the lack of symmetry of the ellipsoidal shape, appear in every step of this work. In order to overcome these

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difficulties we used a technique based on Papkovich–Neuber–Grodski potentials and their interdependence. The proposed technique is applicable to the determination of all coefficients in low-frequency expansions but, as expected, the calculational efforts increase very rapidly with the order of the approximation field. A similar (but simpler) technique was already used for the case of the rigid scatterer [10] and the case of the cavity [11].

The normalized spherical scattering amplitudes and the scattering cross-section are evaluated. This task is reduced to the calculation of certain surface integrals over the surface of the scatterer. The first nonvanishing term of the scattering amplitudes is proportional to the third power of the wave number, while for the scattering cross-section the leading term is proportional to the fourth power of the wave number. The special geometrical cases that correspond to degenerate ellipsoids, such as the sphere, the prolate and the oblate spheroid, the needle and the disc, as well as some physical considerations are also discussed.

2. Statement of the problem

We assume that a plane harmonic wave $\mathbf{u}^{\text{in}}\,\mathrm{e}^{-i\omega t}$ propagating in an infinite, homogeneous isotropic elastic medium V_1 with Lamé constants λ_1, μ_1 and mass density ρ_1 is scattered by a penetrable ellipsoidal body V_2 with Lamé constants λ_2, μ_2, different from λ_1, μ_1, and mass density ρ_2. Suppressing the harmonic time-dependence $\exp\{-i\omega t\}$, where ω is the angular frequency, the incident wave takes the form

$$\mathbf{u}^{\text{in}} = \hat{\mathbf{k}}\,\mathrm{e}^{ik_{p_1}\hat{\mathbf{k}}\cdot\mathbf{r}} \quad \text{or} \quad \mathbf{u}^{\text{in}} = \hat{\mathbf{b}}\,\mathrm{e}^{ik_{s_1}\hat{\mathbf{k}}\cdot\mathbf{r}} \tag{1}$$

for a longitudinal and a transverse wave respectively, where $\hat{\mathbf{k}}$ is the propagation unit vector, $\hat{\mathbf{b}}$ is the polarization unit vector, $\hat{\mathbf{b}}\cdot\hat{\mathbf{k}} = 0$ and k_{p_1} and k_{s_1} are the wavenumbers in V_1 of the P and S wave, respectively.

The scattered field $\mathbf{u}$ as well as the incident wave satisfy the time-independent Navier equation of dynamic elasticity,

$$c_s^2\,\Delta\mathbf{v} + (c_p^2 - c_s^2)\nabla(\nabla\cdot\mathbf{v}) + \omega^2\mathbf{v} = \mathbf{0}\,, \tag{2}$$

where c_p, c_s are the phase velocities for P and S waves, respectively, and $\mathbf{v}$ is the displacement field. The scattered field also satisfies the well-known Kupradze radiation conditions [1].

The boundary conditions on the surface S of the penetrable ellipsoid are

$$\mathbf{u}_1(\mathbf{r}') = \mathbf{u}_2(\mathbf{r}')\,, \qquad T^{(1)}\mathbf{u}_1(\mathbf{r}') = T^{(2)}\mathbf{u}_2(\mathbf{r}')\,, \quad \mathbf{r}' \in S\,, \tag{3}$$

where $\mathbf{u}_i$, $i = 1, 2$, are the total displacement fields for the spaces V_i and $T^{(i)}$ are the surface-stress operators which are given by the expression

$$T^{(i)} = 2\mu_i\hat{\mathbf{n}}\cdot\nabla + \lambda_i\hat{\mathbf{n}}\nabla\cdot + \mu_i\hat{\mathbf{n}}\times\nabla\times \tag{4}$$

and $\hat{\mathbf{n}}$ is the unit normal on S with direction from V_2 to V_1.

The solution $\mathbf{u}_1$ of the above problem has the integral representation [12]

$$\mathbf{u}_1(\mathbf{r}) = \mathbf{u}^{\text{in}}(\mathbf{r}) + \frac{1}{4\pi\rho_1}\left\{\frac{c_{s_1}^2 - c_{s_2}^2}{c_{s_1}^2}\,\omega^2 \int_{v_2} \mathbf{u}_2(\mathbf{r}')\cdot\tilde{\Gamma}^{(1)}(\mathbf{r},\mathbf{r}')\,\mathrm{d}v(\mathbf{r}')\right.$$

$$+\frac{c_{s_1}^2(c_{p_2}^2 - c_{s_2}^2) - c_{s_2}^2(c_{p_1}^2 - c_{s_1}^2)}{c_{s_1}^2}\int_{v_2} \mathbf{u}_2(\mathbf{r}')\cdot\nabla_{r'}\nabla_{r'}\cdot\tilde{\Gamma}^{(1)}(\mathbf{r},\mathbf{r}')\,\mathrm{d}v(\mathbf{r}')$$

$$\left.+\mu_1\int_s \mathbf{u}_2(\mathbf{r}')\cdot[T_{r'}^{(1)} - T_{r'}^{(2)}]\tilde{\Gamma}^{(1)}(\mathbf{r},\mathbf{r}')\,\mathrm{d}s(\mathbf{r}')\right\}, \quad \mathbf{r}\in V_1, \tag{5}$$

where the index r' means calculations with respect to the variable $\mathbf{r}'$ and $\tilde{\Gamma}^{(1)}$* is the fundamental dyadic for $k_p = k_{p_1}$ which is the solution of the equation

$$[c_s^2\Delta_{r'} + (c_p^2 - c_s^2)\nabla_{r'}(\nabla_{r'}\cdot) + \omega^2]\tilde{\Gamma}(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')\tilde{\mathbb{I}}\,. \tag{6}$$

$\tilde{\mathbb{I}}$ is the identity dyadic and δ is the Dirac measure concentrated at $\mathbf{r}$.

The normalized scattering amplitudes for the case of the penetrable scatterer are given by the equations [12]

$$g_r(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \frac{\mathrm{i}k_{p_1}^3(c_{p_1}^2 - c_{p_2}^2)}{4\pi(\lambda_1 + 2\mu_1)}\int_{v_2}\mathbf{u}_2(\mathbf{r}')\,\mathrm{e}^{-\mathrm{i}k_{p_1}\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\mathrm{d}v(\mathbf{r}')\cdot\hat{\mathbf{r}}$$

$$+\frac{k_{p_1}^2}{4\pi(\lambda_1 + 2\mu_1)}\int_s \{\mathbf{u}_2(\mathbf{r}')\otimes\hat{\mathbf{n}}'\} : (\lambda_1 - \lambda_2)\tilde{\mathbb{I}} + 2(\mu_1 - \mu_2)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\}\,\mathrm{e}^{-\mathrm{i}k_{p_1}\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\mathrm{d}s(\mathbf{r}')\,, \tag{7}$$

$$g_\theta(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \frac{\mathrm{i}k_{s_1}^3(c_{s_1}^2 - c_{s_2}^2)}{4\pi\mu_1}\int_{v_2}\mathbf{u}_2(\mathbf{r}')\cdot\hat{\boldsymbol{\theta}}\,\mathrm{e}^{-\mathrm{i}k_{s_1}\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\mathrm{d}v(\mathbf{r}')$$

$$+\frac{k_{s_1}^2(\mu_1 - \mu_2)}{4\pi\mu_1}\int_s \{2\{\mathbf{u}_2(\mathbf{r}')\otimes\hat{\mathbf{n}}'\} : (\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}}) + \{\mathbf{u}_2(\mathbf{r}')\times\hat{\mathbf{n}}'\}\cdot\hat{\boldsymbol{\varphi}}\}\,\mathrm{e}^{-\mathrm{i}k_{s_1}\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\mathrm{d}s(\mathbf{r}')\,, \tag{8}$$

$$g_\varphi(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \frac{\mathrm{i}k_{s_1}^3(c_{s_1}^2 - c_{s_2}^2)}{4\pi\mu_1^2}\int_{v_2}\mathbf{u}_2(\mathbf{r}')\cdot\hat{\boldsymbol{\varphi}}\,\mathrm{e}^{-\mathrm{i}k_{s_1}\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\mathrm{d}v(\mathbf{r}')$$

$$+\frac{k_{s_1}^2(\mu_1 - \mu_2)}{4\pi\mu_1}\int_s \{2\{\mathbf{u}_2(\mathbf{r}')\otimes\hat{\mathbf{n}}'\} : (\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\varphi}}) - \{\mathbf{u}_2(\mathbf{r}')\otimes\hat{\mathbf{n}}\}\cdot\hat{\boldsymbol{\theta}}\}\,\mathrm{e}^{-\mathrm{i}k_{s_1}\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\mathrm{d}s(\mathbf{r}') \tag{9}$$

where the indicated double inner product is defined as

$$(\mathbf{a}\otimes\mathbf{b}) : (\mathbf{c}\otimes\mathbf{d}) = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,. \tag{10}$$

The scattering cross-section is a measure of the disturbance caused by the scatterer to the incident wave. The scattering cross-sections σ_p and σ_s corresponding to an incident P or S wave are expressed by the equations [9]

$$\sigma_p = k_{p_1}\int_{|\hat{\mathbf{r}}|=1}[k_{p_1}^{-3}|g_r(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2 + k_{s_1}^{-3}(|g_\vartheta(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2 + |g_\varphi(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2)]\,\mathrm{d}\Omega(\hat{\mathbf{r}})\,, \tag{11}$$

* The symbol "~" on top of a capital letter denotes a dyadic (second-rank tensor).

$$\sigma_s = k_{s_1} \int_{|\hat{\mathbf{r}}|=1} [k_{p_1}^{-3} |g_r(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2 + k_{s_1}^{-3} (|g_\vartheta(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2 + |g_\varphi(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2)] \, \mathrm{d}\Omega(\hat{\mathbf{r}}) \tag{12}$$

where the integration is taken over the unit sphere.

3. Low-frequency expansions

We consider the total displacement fields, which can be expanded in a convergent power series of the wavenumber,

$$\mathbf{u}_1(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}k_{p_1})^n}{n!} \mathbf{\Phi}_n^{(1)}(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}k_1\tau_1)^n}{n!} \mathbf{\Phi}_n^{(1)}(\mathbf{r}) , \tag{13}$$

$$\mathbf{u}_2(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}k_1\tau_1)^n}{n!} \mathbf{\Phi}_n^{(2)}(\mathbf{r}) \tag{14}$$

where $\mathbf{\Phi}_n^{(1)}$ is the n-th order coefficient for the exterior problem, $\mathbf{\Phi}_n^{(2)}$ is the corresponding coefficient for the interior problem, $k_{s_1} = k_1$ and

$$\tau_i^2 = \frac{\mu_i}{\lambda_i + 2\mu_i} . \tag{15}$$

The coefficients $\mathbf{\Phi}_n^{(i)}$ satisfy the boundary-value problems

$$\begin{gathered} \tau_i^2 \, \Delta\mathbf{\Phi}_n^{(i)}(\mathbf{r}) + (1-\tau_i^2)\nabla(\nabla\cdot\mathbf{\Phi}_n^{(i)}(\mathbf{r})) - n(n-1)q_i\mathbf{\Phi}_{n-2}^{(i)}(\mathbf{r}) = \mathbf{0} , \quad n = 0, 1, 2, \ldots , \\ \mathbf{\Phi}_n^{(1)}(\mathbf{r}') = \mathbf{\Phi}_n^{(2)}(\mathbf{r}') , \qquad T^{(1)}\mathbf{\Phi}_n^{(1)}(\mathbf{r}') = T^{(2)}\mathbf{\Phi}_n^{(2)}(\mathbf{r}') , \quad \mathbf{r}' \in S , \end{gathered} \tag{16}$$

where

$$q_i = \begin{cases} 1 & \text{for } i = 1 , \\ c_{p_1}/c_{p_2} & \text{for } i = 2 . \end{cases} \tag{17}$$

The integral representation for the n-th order coefficient $\mathbf{\Phi}_n^{(1)}$ is

$$\begin{aligned} \mathbf{\Phi}_n^{(1)}(\mathbf{r}) = \frac{1}{4\pi\rho_1} \sum_{\rho=0}^{n} \binom{n}{\rho} \Bigg\{ & \frac{c_{s_1}^2 - c_{s_2}^2}{c_{s_1}^2} \int_{v_2} \mathbf{\Phi}_\rho^{(2)}(\mathbf{r}') \, \frac{(-1)(n-\rho)(n-\rho-1)}{\tau_1^{n-\rho}} \, |\mathbf{r}-\mathbf{r}'|^{n-\rho-3} \\ & \cdot \left\{ \left(1 + \frac{\tau_1^{n-\rho} - 1}{n-\rho}\right)\tilde{\mathbb{I}} + (n-\rho-3) \, \frac{\tau_1^{n-\rho} - 1}{n-\rho} \, \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \otimes \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \right\} \mathrm{d}v(\mathbf{r}') \\ & + \frac{c_{s_1}^2(c_{p_2}^2 - c_{s_2}^2) - c_{s_2}^2(c_{p_1}^2 - c_{s_1}^2)}{c_{s_1}^2} \int_{v_2} \mathbf{\Phi}_\rho^{(2)}(\mathbf{r}')\tau_1^2(n-\rho-1) \\ & \cdot |\mathbf{r}-\mathbf{r}'|^{n-\rho-3} \left\{ (n-\rho-3) \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \otimes \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} + \tilde{\mathbb{I}} \right\} \mathrm{d}v(\mathbf{r}') \\ & + \int_S \mathbf{\Phi}_\rho^{(2)}(\mathbf{r}') \left[-\frac{n-\rho-1}{\tau_1^{n-\rho}(n-\rho+2)} \, |\mathbf{r}-\mathbf{r}'|^{n-\rho-2} \Big\{ (\mu_1 - \mu_2)(2\tau_1^{n-\rho+2} \right. \end{aligned}$$

$$+ (n-\rho))\left[\hat{\mathbf{n}}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbb{I}} + \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \otimes \hat{\mathbf{n}}'\right] + 2(\mu_1 - \mu_2)(n-\rho-3)(\tau_1^{n-\rho+2} - 1)\hat{\mathbf{n}}'$$

$$\cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \otimes \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} + [(\lambda_1 - \lambda_2)(n-\rho-2)\tau_1^{n-\rho+2} + 2(\mu_1 - \mu_2)$$

$$\cdot (\tau_1^{n-\rho+2} - 1)]\hat{\mathbf{n}}' \otimes \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\Big\}\Big]\, ds(\mathbf{r}')\Big\} + \mathbf{a}_n(\hat{\mathbf{k}} \cdot \mathbf{r})^n \tag{18}$$

where

$$\mathbf{a}_n = \begin{cases} \hat{\mathbf{k}}\,, & \text{for P-incident}\,, \\ \dfrac{\hat{\mathbf{b}}}{\tau_1^n}\,, & \text{for S-incident}\,. \end{cases} \tag{19}$$

The asymptotic representation, as $r \to \infty$, for the n-th order coefficient $\mathbf{\Phi}_n^{(1)}$ can be derived from the integral representation (18) if we omit the n-th term in the right-hand side sum, which is of the order of $1/r$.

The low-frequency expansions for the normalized scattering amplitudes are

$$g_r(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = \frac{c_{p_1}^2 - c_{p_2}^2}{4\pi(\lambda_1 + 2\mu_1)} \sum_{n=0}^{\infty} \frac{(ik_1\tau_1)^{n+3}}{n!} \sum_{\rho=0}^{n} \binom{n}{\rho}(-1)^\rho \int_{v_2} \mathbf{\Phi}_{n-\rho}^{(2)}(\mathbf{r}') \cdot (\hat{\mathbf{r}} \cdot \mathbf{r}')^\rho\, dv(\mathbf{r}') \cdot \hat{\mathbf{r}}$$

$$- \frac{1}{4\pi(\lambda_1 + 2\mu_1)} \sum_{n=1}^{\infty} \frac{(ik_1\tau_1)^{n+2}}{n!} \sum_{\rho=0}^{n} \binom{n}{\rho}(-1)^\rho \int_{s} \{\mathbf{\Phi}_{n-\rho}(\mathbf{r}') \otimes \hat{\mathbf{n}}'\}$$

$$: \{(\lambda_1 - \lambda_2)\tilde{\mathbb{I}} + 2(\mu_1 - \mu_2)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\}(\hat{\mathbf{r}} \cdot \mathbf{r}')^\rho\, ds(\mathbf{r}')\,, \tag{20}$$

$$g_\theta(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = \frac{c_{s_1}^2 - c_{s_2}^2}{4\pi\mu_1} \sum_{n=0}^{\infty} \frac{(ik_1\tau_1)^{n+3}}{n!\,\tau_1} \sum_{\rho=0}^{n} \binom{n}{\rho}\left(-\frac{1}{\tau_1}\right)^\rho \int_{v_2} \mathbf{\Phi}_{n-\rho}^{(2)}(\mathbf{r}') \cdot \hat{\boldsymbol{\theta}} \cdot (\hat{\mathbf{r}} \cdot \mathbf{r}')^\rho\, dv(\mathbf{r}')$$

$$+ \frac{\mu_2 - \mu_1}{4\pi\mu_1} \sum_{n=1}^{\infty} \frac{(ik_1\tau_1)^{n+2}}{n!\,\tau_1^2} \sum_{\rho=0}^{n} \binom{n}{\rho}\left(-\frac{1}{\tau_1}\right)^\rho \int_{s} \{2\{\mathbf{\Phi}_{n-\rho}^{(2)}(\mathbf{r}') \otimes \hat{\mathbf{n}}'\}$$

$$: (\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}}) + \{\mathbf{\Phi}_{n-\rho}^{(2)}(\mathbf{r}') \times \hat{\mathbf{n}}'\} \cdot \hat{\boldsymbol{\varphi}}\}(\hat{\mathbf{r}} \cdot \mathbf{r}')^\rho\, ds(\mathbf{r}')\,, \tag{21}$$

$$g_\varphi(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = \frac{c_{s_1}^2 - c_{s_2}^2}{4\pi\mu_1^2} \sum_{n=0}^{\infty} \frac{(ik_1\tau_1)^{n+3}}{n!\,\tau_1} \sum_{\rho=0}^{n} \binom{n}{\rho}\left(-\frac{1}{\tau_1}\right)^\rho \int_{v_2} \mathbf{\Phi}_{n-\rho}^{(2)}(\mathbf{r}') \cdot \hat{\boldsymbol{\varphi}} \cdot (\mathbf{r} \cdot \mathbf{r}')^\rho\, dv(\mathbf{r}')$$

$$+ \frac{\mu_2 - \mu_1}{4\pi\mu_1} \sum_{n=1}^{\infty} \frac{(ik_1\tau_1)^{n+2}}{n!\,\tau_1^2} \sum_{\rho=0}^{n} \binom{n}{\rho}\left(-\frac{1}{\tau_1}\right)^\rho \int_{s} \{2\{\mathbf{\Phi}_{n-\rho}^{(2)}(\mathbf{r}') \otimes \hat{\mathbf{n}}'\}$$

$$: (\hat{\mathbf{r}} \otimes \hat{\boldsymbol{\varphi}}) - \{\mathbf{\Phi}_{n-\rho}^{(2)}(\mathbf{r}') \times \hat{\mathbf{n}}'\} \cdot \hat{\boldsymbol{\theta}}\}(\hat{r} \cdot r')^\rho\, ds(\mathbf{r}')\,. \tag{22}$$

4. The Papkovich potentials in ellipsoidal geometry

It is well known that the solution of the Navier equation in linear elasticity has a representation, due to Grodski–Papkovich–Neuber, involving one vector and one scalar potential [14]. It is also well known that for the representation of the solution of the time-independent Navier equation in the absence of body forces only the vector Papkovich potential suffices and, if we use both the vector and the scalar potentials, there is an interdependence between them.

In our case the sequence of problems to which the scattering problem reduces, is described by the inhomogeneous equation (16). But after long calculations it can be shown that the nonvanishing part of the asymptotic expression of (18) satisfies equation (16), that is, it is a particular solution of (16), and so we can write the solution $\mathbf{\Phi}_n^{(1)}(\mathbf{r})$ of (16) as

$$\mathbf{\Phi}_n^{(1)}(\mathbf{r}) = \mathbf{G}_n^{(1)}(\mathbf{r}) + \mathbf{P}_n^{(1)}(\mathbf{r}) \tag{23}$$

where $\mathbf{P}_n^{(1)}$ is the nonvanishing part, that is the particular solution of (16), and $\mathbf{G}_n^{(1)} = O(1/r)$, for $r \rightarrow \infty$, is the solution of the homogeneous equation

$$\tau_1^2 \Delta \mathbf{G}_n^{(1)}(\mathbf{r}) + (1 - \tau_1^2)\nabla(\nabla \cdot \mathbf{G}_n^{(1)}(\mathbf{r})) = \mathbf{0} \tag{24}$$

with boundary conditions

$$\begin{aligned} \mathbf{G}_n^{(1)}(\mathbf{r}') &= \mathbf{\Phi}_n^{(2)}(\mathbf{r}') - \mathbf{P}_n^{(1)}(\mathbf{r}') , \quad \mathbf{r}' \in S , \\ T^{(1)}\mathbf{G}_n^{(1)}(\mathbf{r}') &= T^{(2)}\mathbf{\Phi}_n^{(2)}(\mathbf{r}') - T^{(1)}\mathbf{P}_n^{(1)}(\mathbf{r}') , \quad \mathbf{r}' \in S . \end{aligned} \tag{25}$$

Using Grodski–Papkovich–Neuber potentials [14] we can express the solution of (24) through the representation

$$\mathbf{G}_n^{(1)}(\mathbf{r}) = \mathbf{A}_n^{(1)}(\mathbf{r}) + \frac{1}{2}(\tau_1^2 - 1)\nabla(\mathbf{r} \cdot \mathbf{A}_n^{(1)}(\mathbf{v}) + B_n^{(1)}(\mathbf{v})) , \quad n = 0, 1, 2, \ldots , \tag{26}$$

where $\mathbf{A}_n^{(1)}$ and $B_n^{(1)}$ are solutions of the homogeneous equations

$$\Delta \mathbf{A}_n^{(1)} = \mathbf{0} , \qquad \Delta B_n^{(1)} = 0 , \quad n = 0, 1, 2, \ldots . \tag{27}$$

Note that the introduction of the scalar potential $B_n^{(1)}$ in the representation (26) is not necessary because of the homogeneity of the equation (24), [14]. We introduce this potential because it constitutes a crucial point (as we will see later) of the technique which we use in order to overcome the difficulties arising in the evaluation of the solution, and in order to be able to reduce the calculation to a finite number of steps. This technique actually describes how the low-frequency approximations can be evaluated in exact closed form by use of harmonic functions alone.

In a similar way we can have a representation for the interior field $\mathbf{\Phi}_n^{(2)}$, using the Papkovich potentials and the corresponding particular solution of the inhomogeneous equation which is satisfied by the interior displacement field.

We assume, now, the scatterer of our problem to be a triaxial ellipsoid. So we have

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \leqq 1 , \qquad 0 < a_3 < a_2 < a_1 < +\infty .$$

In order to reflect the geometrical peculiarities of the scatterer we introduce ellipsoidal harmonic functions. We can use harmonic functions because, as we have seen above, by using Papkovich potentials we reduce the evaluation of the solution of the scattering problem to the evaluation of the coefficients of appropriately chosen expansions of harmonic functions. The ellipsoidal harmonic functions form a complete system of eigenfunctions. We give certain definitions about ellipsoidal harmonics in the Appendix. For details about ellipsoidal harmonics, we refer to Hobson [15].

It is known that the functions

$$\{E_n^m(\mu)E_n^m(\nu):\ n=0,1,\dots,\ m=1,\dots,2n+1\}\,, \tag{28}$$

where E_n^m are the Lamé functions of the first kind, form a complete orthogonal set of surface harmonics on the surface of the ellipsoid.

The vector and scalar Papkovich potentials $\mathbf{A}_n^{(1)}$, $B_n^{(1)}$ in the representation (26) have the following expansions in terms of exterior ellipsoidal harmonics:

$$\mathbf{A}_n^{(1)}(\mathbf{r})=\sum_{k=0}^{\infty}\sum_{m=1}^{2k+1}\mathbf{a}_k^{(1)n,m}\mathbb{F}_k^m(\rho,\mu,\nu)\,, \tag{29}$$

$$B_n^{(1)}(\mathbf{r})=\sum_{k=0}^{\infty}\sum_{m=1}^{2k+1}b_k^{(1)n,m}\mathbb{F}_k^m(\rho,\mu,\nu)\,. \tag{30}$$

The particular solution $\mathbf{P}_n^{(1)}(\mathbf{r})$, as we can observe from its integral form given by (18), has an expansion in terms of surface ellipsoidal harmonics up to degree n. Since on the surface of the scatterer the displacement fields have to become equal, we conclude that the expansion in terms of ellipsoidal harmonics for $\mathbf{G}_n^{(1)}(\mathbf{r})$ must also be up to degree n. This implies that all the solid harmonics in $\mathbf{G}_n^{(1)}(\mathbf{r})$ for $k>n$ must have coefficients equal to zero. Hence, from the vanishing of all these coefficients, the expansion for $\mathbf{\Phi}_n^{(1)}(\mathbf{r})$ in terms of ellipsoidal harmonics degenerates to a finite sum.

Using the relation

$$\nabla(\mathbf{r}\cdot\mathbf{A}_n^{(1)})=\mathbf{A}_n^{(1)}+\sum_{k=0}^{n}\sum_{m=1}^{2k+1}(\mathbf{r}\cdot\mathbf{a}_k^{(1)n,m})\nabla\mathbb{F}_k^m(\rho,\mu,\nu)\,, \tag{31}$$

the representation (26) yields

$$\begin{aligned}\mathbf{G}_n^{(1)}(\mathbf{r})&=\frac{1}{2}(\tau_1^2+1)\sum_{k=0}^{n}\sum_{m=1}^{2k+1}\mathbf{a}_k^{(1)n,m}\mathbb{F}_k^m(\rho,\mu,\nu)\\&\quad+\frac{1}{2}(\tau_1^2-1)\sum_{k=0}^{n}\sum_{m=1}^{2k+1}(\mathbf{r}\cdot\mathbf{a}_k^{(1)n,m}+b_k^{(1)n,m})\nabla\mathbb{F}_k^m(\rho,\mu,\nu)\end{aligned} \tag{32}$$

where

$$\begin{aligned}\nabla\mathbb{F}_k^m(\rho,\mu,\nu)&=(2k+1)\nabla\mathbb{E}_k^m(\rho,\mu,\nu)I_k^m(\rho)\\&\quad-(2k+1)\frac{\hat{\boldsymbol{\rho}}}{h_\rho}\,\frac{\mathbb{E}_k^m(\rho,\mu,\nu)}{[E_k^m(\rho)]^2\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}\,,\end{aligned} \tag{33}$$

$$h_\rho=\frac{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}{\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}\,, \tag{34}$$

is the square root of the ellipsoidal metric coefficient that corresponds to the variable ρ, $h_2^2 = a_1^2 - a_3^2$ and $h_3^2 = a_1^2 - a_2^2$ are the squares of the two semifocal distances and $\hat{\boldsymbol{\rho}}$, the unit curvilinear vector relative to the variable ρ, is given by

$$\hat{\boldsymbol{\rho}} = \frac{\rho}{h_\rho} \sum_{i=1}^{3} \frac{x_i}{\rho^2 - a_1^2 + a_i^2} \hat{\mathbf{x}}_i \tag{35}$$

where $\hat{\mathbf{x}}_i$ are the cartesian base vectors.

Inserting (32), (33) into (23) we conclude that

$$\begin{aligned}\boldsymbol{\Phi}_n^{(1)}(\mathbf{r}) = \frac{1}{2} \Big\{ &(\tau_1^2 + 1) \sum_{k=0}^{n} \sum_{m=1}^{2k+1} \mathbf{a}_k^{(1)n,m}(2k+1) I_k^m(\rho) \mathbb{E}_k^m(\rho, \mu, \nu) \\ &+ (\tau_1^2 - 1) \sum_{k=0}^{n} \sum_{m=1}^{2k+1} (\mathbf{r} \cdot \mathbf{a}_k^{(1)n,m})(2k+1) I_k^m(\rho) \nabla \mathbb{E}_k^m(\rho, \mu, \nu) \\ &+ (\tau_1^2 - 1) \sum_{k=0}^{n+1} \sum_{m=1}^{2k+1} b_k^{(1)n,m}(2k+1) I_k^m(\rho) \nabla \mathbb{E}_k^m(\rho, \mu, \nu) \Big\} \\ &- \frac{1}{2}(\tau_1^2 - 1) \frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} \Big\{ \sum_{k=0}^{n} \sum_{m-1}^{2k+1} (\mathbf{r} \cdot \mathbf{a}_k^{(1)n,m})(2k+1) \frac{\mathbb{E}_k^m(\rho, \mu, \nu)}{\{E_k^m(\rho)\}^2} \\ &+ \sum_{k=0}^{n+1} \sum_{m=1}^{2k+1} b_k^{(1)n,m}(2k+1) \frac{\mathbb{E}_k^m(\rho, \mu, \nu)}{\{E_k^m(\rho)\}^2} \Big\} + \mathbf{P}_n^{(1)}(\mathbf{r}) .\end{aligned} \tag{36}$$

5. The zeroth and the first-order approximation of the displacement fields

We will now propose a technique to solve the zeroth and the first-order approximations of the displacement fields in the low-frequency region. It has been proved by Jones [13] that these terms give us enough information for the total field.

The zeroth-order approximation of the displacement field is the solution of the boundary-value problem

$$\begin{aligned} &\tau_i^2 \, \Delta \boldsymbol{\Phi}_0^{(i)}(\mathbf{r}) + (1 - \tau_i^2) \nabla(\nabla \cdot \boldsymbol{\Phi}_0^{(i)}(\mathbf{r})) = \mathbf{0} , \quad \mathbf{r} \in V_i , \quad i = 1, 2 , \\ &\boldsymbol{\Phi}_0^{(1)}(\mathbf{r}') = \boldsymbol{\Phi}_0^{(2)}(\mathbf{r}') , \quad \mathbf{r}' \in S , \\ &T^{(1)} \boldsymbol{\Phi}_0^{(1)}(\mathbf{r}') = T^{(2)} \boldsymbol{\Phi}_0^{(2)}(\mathbf{r}') , \quad \mathbf{r}' \in S , \\ &\boldsymbol{\Phi}_0^{(1)}(\mathbf{r}) = \mathbf{a}_0 + O\left(\frac{1}{r}\right) \end{aligned} \tag{37}$$

where $\mathbf{a}_0$ is given by (19), while the first-order approximation is the solution of the boundary value problem

$$\tau_i \, \Delta \boldsymbol{\Phi}_1^{(i)}(\mathbf{r}) + (1 - \tau_i^2) \nabla(\nabla \cdot \boldsymbol{\Phi}_1^{(i)}(\mathbf{r})) = \mathbf{0} , \quad \mathbf{r} \in V_i , \quad i = 1, 2 ,$$

$$\boldsymbol{\Phi}_1^{(1)}(\mathbf{r}') = \boldsymbol{\Phi}_1^{(2)}(\mathbf{r}') , \quad \mathbf{r}' \in S ,$$

$$T^{(1)}\mathbf{\Phi}_1^{(1)}(\mathbf{r}') = T^{(2)}\mathbf{\Phi}_1^{(2)}(\mathbf{r}') , \quad \mathbf{r}' \in S ,$$

$$\mathbf{\Phi}_1^{(1)}(\mathbf{r}) = \mathbf{a}_1(\hat{\mathbf{k}}\cdot\mathbf{r}) + \mathrm{O}\left(\frac{1}{r}\right) \tag{38}$$

where $\mathbf{a}_1$ is given by (19).

Substituting in (36) for $n = 0$ we have the following representation of $\mathbf{\Phi}_0^{(1)}(\mathbf{r})$:

$$\begin{aligned}\mathbf{\Phi}_0^{(1)}(\mathbf{r}) &= \frac{1}{2}(\tau_1^2+1)\mathbf{a}_0^{(1)0,1}I_0^1(\rho) + \frac{3}{2}(\tau_1^2-1)h_1h_2h_3\sum_{m=1}^{3}\frac{b_1^{(1)0,m}}{h_m}I_1^m(\rho)\hat{\mathbf{x}}_m \\ &\quad - \frac{1}{2}(\tau_1^2-1)\frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\left\{b_0^{(1)0,1} + \sum_{m=1}^{3}\left(\frac{a_{0,m}^{(1)0,1}h_m}{h_1h_2h_3}\right.\right. \\ &\quad \left.\left. + \frac{3b_1^{(1)0,m}}{\{E_1^m(\rho)\}^2}\right)\mathbb{E}_1^m(\rho,\mu,\nu)\right\} + \mathbf{a}_0 .\end{aligned} \tag{39}$$

For the zeroth-order approximation of the interior field we have that

$$\mathbf{\Phi}_0^{(2)}(\mathbf{r}) = \frac{1}{2}(\tau_2^2+1)\mathbf{a}_0^{(2)0,1}\mathbb{E}_0^1(\rho,\mu,\nu) . \tag{40}$$

The first-order approximations for the exterior and the interior fields have the representations

$$\begin{aligned}\mathbf{\Phi}_1^{(1)}(r) &= \frac{\mathbf{a}_1}{h_1h_2h_3}\sum_{m=1}^{3}k_mh_m\mathbb{E}_1^m(\rho,\mu,\nu) \\ &\quad + \frac{1}{2}(\tau_1^2+1)\left[\mathbf{a}_0^{(1)1,1}I_0^1(\rho) + \sum_{m=1}^{3}\mathbf{a}_1^{(1)1,m}3I_1^m(\rho)\mathbb{E}_1^m(\rho,\mu,\nu)\right] \\ &\quad + \frac{1}{2}(\tau_1^2-1)\left[\sum_{m=1}^{3}\left(\sum_{k=1}^{3}3\frac{h_m}{h_k}a_{1m}^{(1)1,k}I_1^{(k)}(\rho)\hat{\mathbf{x}}_k\right)\mathbb{E}_1^m(\rho,\mu,\nu)\right. \\ &\quad + 3\sum_{m=1}^{3}b_1^{(1)1,m}I_1^m(\rho)\frac{h_1h_2h_3}{h_m}\hat{\mathbf{x}}_m + 10b_2^{(1)1,1}(\Lambda-a_1^2)(\Lambda-a_2^2)(\Lambda-a_3^2)I_2^1(\rho) \\ &\quad \cdot\sum_{m=1}^{3}\frac{h_m\hat{\mathbf{x}}_m}{h_1h_2h_3(\Lambda-a_m^2)}\mathbb{E}_1^m(\rho,\mu,\nu) + 10b_2^{(1)1,2}(\Lambda'-a_1^2)(\Lambda'-a_2^2)(\Lambda'-a_3^2)I_2^2(\rho) \\ &\quad \cdot\sum_{m=1}^{3}\frac{h_m\hat{\mathbf{x}}_m}{h_1h_2h_3(\Lambda'-a_m^2)}\mathbb{E}_1^m(\rho,\mu,\nu) \\ &\quad + 5h_1h_2h_3\left[b_2^{(1)1,3}I_2^3(\rho)\frac{\hat{\mathbf{x}}_2}{h_2} + b_2^{(1)1,4}I_2^4(\rho)\frac{\hat{\mathbf{x}}_3}{h_3}\right]\mathbb{E}_1^1(\rho,\mu,\nu) \\ &\quad + 5h_1h_2h_3\left[b_2^{(1)1,3}I_2^3(\rho)\frac{\hat{\mathbf{x}}_1}{h_1} + b_2^{(1)1,5}I_2^5(\rho)\frac{\hat{\mathbf{x}}_3}{h_3}\right]\mathbb{E}_1^2(\rho,\mu,\nu) \\ &\quad + 5h_1h_2h_3\left[b_2^{(1)1,4}I_2^4(\rho)\frac{\hat{\mathbf{x}}_1}{h_1} + b_2^{(1)1,5}I_2^5(\rho)\frac{\hat{\mathbf{x}}_2}{h_2}\right]\mathbb{E}_1^3(\rho,\mu,\nu) \\ &\quad + \frac{1}{2}(\tau_1^2-1)\frac{\hat{\boldsymbol{\rho}}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\left\{h_1h_2h_3\sum_{k=1}^{3}\frac{1}{h_k}a_{1k}^{(1)1,k} + b_0^{(1)1,1}\right.\end{aligned}$$

$$+\sum_{n=1}^{3}\left[\frac{h_n}{h_1h_2h_3}\,a_{0n}^{(1)1,1}+3b_1^{(1)1,n}\frac{1}{(E_1^n(\rho))^2}\right]\mathbb{E}_1^n(\rho,\mu,\nu)$$

$$+\left[\frac{5b_2^{(1)1,1}}{E_2^1(\rho)}-\frac{3}{h_1h_2h_3(\Lambda-\Lambda')}\{a_{11}^{(1)1,1}h_1(\Lambda'-a_1^2)\right.$$

$$\left.-a_{12}^{(1)1,2}h_2(\Lambda'-a_2^2)+a_{13}^{(1)1,3}h_3(\Lambda'-a_3^2)\}\right]E_2^1(\mu)E_2^1(\nu)$$

$$+\left[\frac{5b_2^{(1)1,2}}{E_2^2(\rho)}+\frac{3}{h_1h_2h_3(\Lambda-\Lambda')}\{a_{11}^{(1)1,1}h_1(\Lambda-a_1^2)\right.$$

$$\left.-a_{12}^{(1)1,2}h_2(\Lambda-a_2^2)+a_{13}^{(1)1,3}h_3(\Lambda-a_3^2)\}\right]E_2^2(\mu)E_2^2(\nu)$$

$$+\left[\frac{5b_2^{(1)1,3}}{(E_2^3(\rho))^2}+\frac{3}{h_3}\left(\frac{a_{12}^{(1)1,1}}{h_1(E_1^1(\rho))^2}+\frac{a_{11}^{(1)1,2}}{h_2(E_1^2(\rho))^2}\right)\right]\mathbb{E}_2^3(\rho,\mu,\nu)$$

$$+\left[\frac{5b_2^{(1)1,4}}{(E_2^4(\rho))^2}+\frac{3}{h_2}\left(\frac{a_{13}^{(1)1,1}}{h_1(E_1^1(\rho))^2}+\frac{a_{11}^{(1)1,3}}{h_3(E_1^3(\rho))^2}\right)\right]\mathbb{E}_2^4(\rho,\mu,\nu)$$

$$+\left[\frac{5b_2^{(1)1,5}}{(E_2^5(\rho))^2}+\frac{3}{h_1}\left(\frac{a_{12}^{(1)1,3}}{h_3(E_1^3(\rho))^2}+\frac{a_{13}^{(1)1,2}}{h_2(E_1^2(\rho))^2}\right)\right]\mathbb{E}_2^5(\rho,\mu,\nu)\Bigg]\Bigg\} \tag{41}$$

and

$$\mathbf{\Phi}_1^{(2)}(\mathbf{r})=\frac{1}{2}(\tau_2^2+1)\left\{\mathbf{a}_0^{(2)1,1}+\sum_{m=1}^{3}\mathbf{a}_1^{(2)1,m}\mathbb{E}_1^m(\rho,\mu,\nu)\right\}$$

$$+\frac{1}{2}(\tau_2^2-1)\sum_{m=1}^{3}\left(\sum_{k=1}^{3}\frac{h_m}{h_k}\,a_{1m}^{(2)1,k}\hat{\mathbf{x}}_k\right)\mathbb{E}_1^m(\rho,\mu,\nu)\,. \tag{42}$$

The expressions for $\mathbf{\Phi}_0^{(1)}(\mathbf{r})$, $\mathbf{\Phi}_1^{(1)}(\mathbf{r})$ as given by (39), (41) are the sum of two parts. One can be expanded in a finite sum of ellipsoidal harmonics, to which we will refer in the following as the "cartesian" part of the solution, and another one which is multiplied by the factor $\hat{\boldsymbol{\rho}}\cdot\{(\rho^2-\mu^2)(\rho^2-\nu^2)\}^{-1/2}$. This part can not be expressed in terms of a finite expansion of ellipsoidal harmonics, because of the existence of this factor. We will refer to this part as the "ellipsoidal" part of the solution. The expressions for $\mathbf{\Phi}_0^{(2)}(\mathbf{r})$, $\mathbf{\Phi}_1^{(2)}(\mathbf{r})$ are only of "cartesian" type. This difference between the forms of $\mathbf{\Phi}_n^{(1)}(\mathbf{r})$ and $\mathbf{\Phi}_n^{(2)}(\mathbf{r})$, $n=0,1$, is due to the fact that $\mathbf{\Phi}_n^{(1)}(\mathbf{r})$, as solution of an exterior problem, is expressed in terms of second-kind ellipsoidal harmonics, while $\mathbf{\Phi}_n^{(2)}(\mathbf{r})$, as solution of an interior problem, involves first-kind ellipsoidal harmonics.

Applying now the surface-stress operator to $\mathbf{\Phi}_n^{(i)}$, $n=0,1$, $i=1,2$, we observe that all the terms at the two sides of the equation of the boundary condition that arise from the equality of the traction fields, are multiplied by the factor $a_2a_3(a_1^2-\mu^2)^{-1/2}(a_1^2-\nu^2)^{-\frac{1}{2}}$. The presence of this common factor is due to the expression of the surface-traction operator in ellipsoidal coordinates. Considering $T^{(1)}\mathbf{\Phi}_n^{(1)}$, we have that there are three types of terms. Terms constituting the "cartesian" part, terms involving the factor $\hat{\boldsymbol{\rho}}\cdot\{(\rho^2-\mu^2)(\rho^2-\nu^2)\}^{-\frac{1}{2}}$ which constitute the "ellipsoidal" part and, finally, terms that contain the normal derivative $\partial_\rho\{\hat{\boldsymbol{\rho}}\{(\rho^2-\mu^2)(\rho^2-\nu^2)\}^{-\frac{1}{2}}\}$, which form the "ellipsoidal-derivative" part. When we apply the surface-stress operator to $\mathbf{\Phi}_n^{(2)}(\mathbf{r})$, we have terms of only one type which constitute the "cartesian" part of $T^{(2)}\mathbf{\Phi}_n^{(2)}(\mathbf{r})$.

In order to satisfy the boundary conditions we will employ the special forms for the displacement and the traction fields. In this procedure we will use the interdependence of Papkovich potentials.

We apply the first of the boundary conditions which must be satisfied by the displacement vector in order to be continuous across the boundary. The first difficulty, which arises due to the factor $\hat{\boldsymbol{\rho}}\{(\rho^2-\mu^2)(\rho^2-\nu^2)\}^{-1/2}$, is that the "ellipsoidal" parts of $\mathbf{\Phi}_0^{(1)}(\mathbf{r})$, $\mathbf{\Phi}_1^{(1)}(\mathbf{r})$ do not have finite expansions in terms of surface ellipsoidal harmonics. We choose to express the $b_k^{(1)n,m}$, $n=0$, 1, in terms of $a_k^{(1)n,m}$ in such a way that the term which is multiplied by the factor $\hat{\boldsymbol{\rho}}\{(\rho^2-\mu^2)(\rho^2-\nu^2)\}^{-1/2}$ vanishes on the surface of the scatterer. We have this freedom because of the interdependence of Papkovich potentials. In this step we also use the orthogonality of the surface ellipsoidal harmonics. In order to have continuity of the interface traction, the second of the boundary conditions must be satisfied. At this step we have no more freedom to let terms vanish independently because we have already established the connection between the coefficients b and a. Based on the special form of the traction fields, we choose to let vanish independently the "ellipsoidal" part and the "ellipsoidal-derivative" part of $T^{(1)}\mathbf{\Phi}_0^{(1)}T^{(1)}\mathbf{\Phi}_1^{(1)}$. After long calculations, using many relations connecting the ellipsoidal harmonics, we conclude that the relations which we obtain from the vanishing of the "ellipsoidal" part of $\mathbf{\Phi}_0^{(1)}, \mathbf{\Phi}_1^{(1)}$ are the same as those which we obtain from the vanishing of both the "ellipsoidal" part and the "ellipsoidal-derivative" part of $T^{(1)}\mathbf{\Phi}_0^{(1)}$, $T^{(1)}\mathbf{\Phi}_1^{(1)}$ at the surface of the scatterer. So the trick we have used to remove independently the "ellipsoidal" and the "ellipsoidal-derivative" parts of the traction field gives us relations consistent with the relations that are derived from the first of the boundary conditions. Equating now the "cartesian" parts of $T^{(1)}\mathbf{\Phi}_0^{(1)}$, $T^{(1)}\mathbf{\Phi}_1^{(1)}$ and $T^{(2)}\mathbf{\Phi}_0^{(2)}$, $T^{(2)}\mathbf{\Phi}_1^{(2)}$, respectively, the only actually remaining parts, we obtain new relations between the coefficients b and a. So, it is possible to calculate in a finite number of steps the two approximations for the displacement fields. Here, we want to mention that there are terms in the expressions for the traction fields which "seem" to belong to the "ellipsoidal" part or the "ellipsoidal-derivative" part of the expressions and which are actually of "cartesian" type. Many mathematical manipulations were needed in order to arrive at the correct characterization of the terms.

As a conclusion, it is clear from the above discussion that the key of our method in order to obtain the solution in closed form is the introduction of the scalar Papkovich potential.

Applying the proposed technique in order to solve the boundary-value problems for the first two approximations, we conclude that

$$\mathbf{\Phi}_0^{(1)}(\mathbf{r}) = \mathbf{\Phi}_0^{(2)}(\mathbf{r}) = \mathbf{a}_0\,, \tag{43}$$

and for the first-order fields we have:

For the exterior displacement field

$$\begin{aligned}\mathbf{\Phi}_1^{(1)}(\mathbf{r}) &= \mathbf{a}_1\otimes\hat{\mathbf{k}}\cdot\mathbf{r} + \mathbf{a}_1\otimes\hat{\mathbf{k}} : {}^{(4)}\tilde{Q}_1(\rho)\cdot\mathbf{r}\\ &\quad + \mathbf{a}_1\otimes\hat{\mathbf{k}} : \{\tilde{P}(\rho) + {}^{(4)}\tilde{Q}_2(\rho) : \mathbf{r}\otimes\mathbf{r}\}\otimes\frac{\hat{\boldsymbol{\rho}}}{\{(\rho^2-\mu^2)(\rho^2-\nu^2)\}^{1/2}}\end{aligned} \tag{44}$$

where

$$
\begin{aligned}
{}^{(4)}\tilde{Q}_1(\rho) = \sum_{n=1}^{3} \Bigg[& \sum_{\substack{k=1\\k\neq n}}^{3} \left(\frac{(\tau_1^2+1)}{2} I_1^n(\rho) + \frac{(\tau_1^2-1)}{2} \right) I_1^k(\rho) \tilde{G}_{kn} \otimes \hat{\mathbf{x}}_k + \frac{3\tau_1^2}{D} \frac{I_1^n(\rho)}{h_n} \tilde{J}_n \otimes \hat{\mathbf{x}}_n \\
& + \frac{\tau_1^2-1}{D} \sum_{k=1}^{3} \frac{1}{h_k(\Lambda-\Lambda')} \left\{ \frac{\Lambda(\Lambda-a_1^2)(\Lambda-a_2^2)(\Lambda-a_3^2)}{(\Lambda-a_n^2)(\Lambda-a_k^2)} I_2^1(\rho) \right. \\
& \left. - \frac{\Lambda'(\Lambda'-a_1^2)(\Lambda'-a_2^2)(\Lambda'-a_3^2)}{(\Lambda'-a_n^2)(\Lambda'-a_k^2)} I_2^2(\rho) \right\} \tilde{J}_k \otimes \tilde{\mathbf{x}}_n \Bigg] \otimes \hat{\mathbf{x}}_n \\
& - \frac{\tau_1^2-1}{2} [(a_1^2+a_2^2) I_2^3(\rho) \tilde{G}_{12} \otimes (\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_1) \\
& + (a_1^2+a_3^2) I_2^4(\rho) \tilde{G}_{13} \otimes (\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 + \hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_1) \\
& + (a_2^2+a_3^2) I_2^5(\rho) \tilde{G}_{23} \otimes (\hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_3 + \hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_2)] \,, \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{P}(\rho) = & \frac{(\tau_1^2-1)(\rho^2-a_1^2)}{2D(\Lambda-\Lambda')} \sum_{k=1}^{3} \frac{1}{h_k} \left[\frac{(\Lambda-a_1^2)(\Lambda-a_2^2)(\Lambda-a_3^2)}{\Lambda-a_k^2} \frac{1}{(E_2^1(\rho))^2} \right. \\
& \left. - \frac{(\Lambda'-a_1^2)(\Lambda'-a_2^2)(\Lambda'-a_3^2)}{\Lambda'-a_k^2} \frac{1}{(E_2^2(\rho))^2} \right] \tilde{J}_k \qquad (46)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(4)}\tilde{Q}_2(\rho) = & \frac{(\tau_1^2-1)(\rho^2-a_1^2)}{2D(\Lambda-\Lambda')} \sum_{n=1}^{3} \sum_{k=1}^{3} \frac{1}{h_k} \left[\frac{(\Lambda-a_1^2)(\Lambda-a_2^2)(\Lambda-a_3^2)}{(\Lambda-a_k^2)(\Lambda-a_n^2)} \frac{1}{(E_2^1(\rho))^2} \right. \\
& \left. - \frac{(\Lambda'-a_1^2)(\Lambda'-a_2^2)(\Lambda'-a_3^2)}{(\Lambda'-a_k^2)(\Lambda'-a_n^2)} \frac{1}{(E_2^2(\rho))^2} \right] \tilde{J}_k \otimes \hat{\mathbf{x}}_n \otimes \hat{\mathbf{x}}_n \\
& - (\tau_1^2-1)(\rho^2-a_1^2) \left[\frac{1}{(E_2^3(\rho))^2} \tilde{G}_{12} \otimes \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + \frac{1}{(E_2^4(\rho))^2} \tilde{G}_{13} \otimes \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 \right. \\
& \left. + \frac{1}{(E_2^5(\rho))^2} \tilde{G}_{23} \otimes \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_3 \right] . \qquad (47)
\end{aligned}
$$

In relations (45), (46), (47) D is the determinant whose n, k-entry is given by the relation

$$
\begin{aligned}
d_{nk} = \frac{h_n}{h_k} \Bigg\{ & 3\mu_1 \left(2\tau_1^2 I_1^n(a_1) - \frac{1}{a_1 a_2 a_3} \right) \delta_{nk} + 3\lambda_1 \tau_1^2 I_1^k(a_1) \\
& + \frac{1}{\Lambda-\Lambda'} \left\{ \frac{\Lambda(\Lambda-a_1^2)(\Lambda-a_2^2)(\Lambda-a_3^2)}{(\Lambda-a_n^2)(\Lambda-a_k^2)} I_2^1(a_1) \right. \\
& \left. - \frac{\Lambda'(\Lambda'-a_1^2)(\Lambda'-a_2^2)(\Lambda-a_3^2)}{(\Lambda'-a_n^2)(\Lambda'-a_k^2)} I_2^2(a_1) \right\} \\
& \cdot 2(\tau_1^2-1)[\mu_1 - (\Lambda-\Lambda')\mu_2 \tau_2^2] - 3\tau_1^2 I_1^k(a_1) \tau_2^2 (\lambda_2 - 2\mu_2 \delta_{nk}) \Bigg\} \,, \qquad (48)
\end{aligned}
$$

and $\tilde{J}_k$ is a second-rank tensor given by

$$
\tilde{J}_k = \sum_{n=1}^{3} (-1)^{k+n+1} D_{nk} h_n \{ 2(\mu_1 - \mu_2) \hat{\mathbf{x}}_n \otimes \hat{\mathbf{x}}_n + (\lambda_1 - \lambda_2 \tau_2^2) \tilde{\mathbb{I}} \} \,. \qquad (49)
$$

In (49) D_{nk} is the minor determinant of D corresponding to the n, k-entry. Further, $\tilde{G}_{nk}$ is a second-rank tensor given by the equation

$$\tilde{G}_{nk} = \frac{(\mu_1 - \mu_2\tau_2^2)}{M_{nk}} (\hat{\mathbf{x}}_n \otimes \hat{\mathbf{x}}_k + \hat{\mathbf{x}}_k \otimes \hat{\mathbf{x}}_n) \tag{50}$$

and

$$\begin{aligned} M_{nk} &= a_1a_2a_3 I_1^n(a_1) I_1^k(a_1)\{\mu_2 + 2I_2^2(\mu_1 - \mu_2\tau_2^2)\} \\ &\quad - (I_1^n(a_1) + I_1^k(a_1))\{\mu_2 + \tau_1^2(\mu_1 - \mu_2\tau_2^2)\} + (\tau_1^2 - 1)(\mu_1 - \mu_2\tau_2^2)(a_n^2 + a_k^2) I_2^{n+k}(a_1) \\ &\quad + a_1a_2a_3(a_n^2 I_1^n(a_1) + a_k^2 I_1^k(a_1))\tau^2(\mu_1 - \mu_2\tau_2^2) + \frac{\mu_1}{a_1a_2a_3} . \end{aligned} \tag{51}$$

The interior displacement field

The interior displacement field is given by the relation

$$\mathbf{\Phi}_1^{(2)}(\mathbf{r}) = \mathbf{a}_1 \otimes \hat{\mathbf{k}} : {}^{(4)}\tilde{R} \cdot \mathbf{r} \tag{52}$$

where ${}^{(4)}\tilde{R}$ is a fourth-rank tensor,

$$\begin{aligned} {}^{(4)}\tilde{R} &= \frac{h_1h_2h_3}{2} \sum_{m=1}^{3} \left\{ \sum_{k=1}^{3} \left\{ \frac{1}{h_m} (\tau_2^2 + 1)\tilde{N}_{km} + \frac{1}{h_k} (\tau_2^2 - 1)\tilde{N}_{mk} \right\} \otimes \hat{\mathbf{x}}_k \otimes \hat{\mathbf{x}}_m \right. \\ &\quad \left. + \frac{2\tau_2^2}{h_m} \tilde{M}_m \otimes \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \right\} \end{aligned} \tag{53}$$

in which $\tilde{M}_k$ is a second-rank tensor given by the relation

$$\begin{aligned} \tilde{M}_k &= \frac{1}{h_1h_2h_3} \left\{ h_k \hat{x}_k \otimes \hat{x}_k + 3\tau_1^2 I_1^k(a_1) \frac{\tilde{J}_k}{D} \right. \\ &\quad + \frac{(\tau_1^2 - 1)}{D} \sum_{n=1}^{3} \frac{h_k}{h_n} \left[\frac{\Lambda(\Lambda - a_1^2)(\Lambda - a_2^2)(\Lambda - a_3^2)}{(\Lambda - a_n^2)(\Lambda - a_k^2)} I_2^1(a_1) \right. \\ &\quad \left. \left. - \frac{\Lambda'(\Lambda' - a_1^2)(\Lambda' - a_2^2)(\Lambda' - a_3^2)}{(\Lambda' - a_n^2)(\Lambda' - a_k^2)} I_2^2(a_1) \right] \tilde{J}_n \right\} \end{aligned} \tag{54}$$

and $\tilde{N}_{kn}$ is given by

$$\tilde{N}_{kn} = \frac{h_n}{2\tau_2^2 h_1h_2h_3} \{\hat{\mathbf{x}}_n \otimes \hat{\mathbf{x}}_k - \hat{\mathbf{x}}_k \otimes \hat{\mathbf{x}}_n - 3I_1^k(a_1)\tilde{G}_{nk} + 3I_1^n(a_1)\tilde{G}_{kn}\} . \tag{55}$$

6. Scattering amplitudes and scattering cross-section

From equations (20), (21), (22) the leading-term approximation for the normalized scattering amplitudes can be evaluated as $k \to 0$. Substituting the solution $\mathbf{\Phi}_0^{(2)}(\mathbf{r}) = \mathbf{a}_0$ and using the relation

$$\int_s \hat{\boldsymbol{\eta}} \otimes \mathbf{r}' \, ds(\mathbf{r}') = \int_{v_2} \nabla \mathbf{r}' \, dv(\mathbf{r}') = V_2 \tilde{\mathbb{I}} , \tag{56}$$

we find

$$g_r(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = \frac{\mathrm{i} k_1^3 \tau_1^3}{4\pi(\lambda_1 + 2\mu_1)} \int_s \boldsymbol{\Phi}_1^{(2)}(\mathbf{r}') \otimes \hat{\boldsymbol{\eta}}' \, ds(\mathbf{r}') : \{(\lambda_1 - \lambda_2)\tilde{\mathbb{I}}$$

$$+ 2(\mu_1 - \mu_2)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\} + \frac{\mathrm{i}k_1^3}{4\pi}\left(\frac{\rho_2}{\rho_1} - 1\right) V_2(\mathbf{a}_0 \cdot \hat{\mathbf{r}}) + O(k_1^4) , \tag{57}$$

$$g_\theta(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = -\frac{(\mu_2 - \mu_1)\mathrm{i}k_1^3 \tau_1}{4\pi\mu_1} \left\{ \int_s \{2\boldsymbol{\Phi}_1^{(2)}(\mathbf{r}') \otimes \hat{\boldsymbol{\eta}}'\} : \hat{\mathbf{r}} \otimes \hat{\boldsymbol{\theta}} \right.$$

$$\left. + \{\boldsymbol{\Phi}_1^{(2)}(\mathbf{r}') \times \hat{\boldsymbol{\eta}}'\} \cdot \hat{\boldsymbol{\varphi}} \right\} ds(\mathbf{r}') + \frac{\mathrm{i}k_1^3}{4\pi}\left(\frac{\rho_2}{\rho_1} - 1\right) V_2(\mathbf{a}_0 \cdot \hat{\boldsymbol{\theta}}) + O(k_1^4) , \tag{58}$$

$$g_\varphi(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = -\frac{(\mu_2 - \mu_1)\mathrm{i}k_1^3 \tau_1}{4\pi\mu_1} \left\{ \int_s \{2\boldsymbol{\Phi}_1^{(2)}(\mathbf{r}') \otimes \hat{\boldsymbol{\eta}}'\} : \hat{\mathbf{r}} \otimes \hat{\boldsymbol{\varphi}} \right.$$

$$\left. - \{\boldsymbol{\Phi}_1^{(2)}(\mathbf{r}') \times \hat{\boldsymbol{\eta}}'\} \cdot \hat{\boldsymbol{\theta}} \right\} ds(\mathbf{r}') + \frac{\mathrm{i}k_1^3}{4\pi}\left(\frac{\rho_2}{\rho_1} - 1\right) V_2(\mathbf{a}_0 \cdot \hat{\boldsymbol{\varphi}}) + O(k_1^4) . \tag{59}$$

The leading-term approximation for the scattering cross-section follows from substitution of (57), (58), (59) in (11), (12). After long calculations we obtain the relations

(i) *for P-incidence*

$$\sigma_p = \frac{k_1^4 \tau_1^3 V_2^2}{12\pi} \left[\frac{1}{5} \left\{ \frac{15(\lambda_1 - \lambda_2)^2 + 20(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) + 8(\mu_1 - \mu_2)^2}{(\lambda_1 + 2\mu_1)^2} \tau_1 \right.\right.$$

$$\left. - \frac{8(\mu_2 - \mu_1)^2}{\mu_1^2} \right\} |\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} : \tilde{R}_2|^2 + \frac{2(\mu_2 - \mu_1)^2}{\mu_1^2} |\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} : {}^{(3)}\tilde{R}_1|^2$$

$$\left. + \frac{4(\mu_1 - \mu_2)^2}{5} \left\{ \frac{\tau_1}{(\lambda_1 + 2\mu_1)^2} + \frac{4}{\mu_1^2} \right\} \| \hat{\mathbf{k}} \otimes \hat{\mathbf{k}} : {}^{(4)}\tilde{R} \|^2 \right]$$

$$+ \frac{\tau_1 k_1^4}{12\pi} (\tau_1^3 + 2)\left(\frac{\rho_2}{\rho_1} - 1\right)^2 V_2^2 + O(k_1^6) \tag{60}$$

where ${}^{(3)}R_1$ is the triadic

$${}^{(3)}\tilde{R}_1 = \frac{h_1 h_2 h_3}{2} \sum_{m=1}^{3} \sum_{k=1}^{3} \left\{ \frac{1}{h_m} (\tau_2^2 + 1)\hat{N}_{km} + \frac{1}{h_k} (\tau_2^2 - 1)\tilde{N}_{mk} \right\} \otimes \tilde{x}_k \otimes \tilde{x}_m \tag{61}$$

with $\tilde{R}_2$ given by the relation

$$\tilde{R}_2 = h_1 h_2 h_3 \tau_2^2 \sum_{m=1}^{3} \frac{1}{h_m} \tilde{M}_m ; \tag{62}$$

(ii) *for S-incidence*

$$\sigma_s = \frac{k_1^4 V_2^2}{12\pi}\left[\frac{1}{5}\left\{\frac{15(\lambda_1-\lambda_2)^2+20(\lambda_1-\lambda_2)(\mu_1-\mu_2)+8(\mu_1-\mu_2)^2}{(\lambda_1+2\mu_1)^2}\tau_1\right.\right.$$
$$\left. - \frac{8(\mu_2-\mu_1)^2}{\mu_1^2}\right\}|\hat{\mathbf{b}}\otimes\hat{\mathbf{k}}:\tilde{R}_2|^2 + \frac{2(\mu_2-\mu_1)^2}{\mu_1}|\hat{\mathbf{b}}\otimes\hat{\mathbf{k}}:{}^{(3)}\tilde{R}_1|^2$$
$$\left. + \frac{4(\mu_1-\mu_2)^2}{5}\left\{\frac{\tau_1}{(\lambda_1+2\mu_1)^2}+\frac{4}{\mu_1^2}\right\}\|\hat{\mathbf{b}}\otimes\hat{\mathbf{k}}:{}^{(4)}\tilde{R}\|^2\right]$$
$$+ \frac{\tau_1 k_1^4}{12\pi}(\tau_1^3+2)\left(\frac{\rho_2}{\rho_1}-1\right)^2 V_2^2 + O(k_1^6) \tag{63}$$

where the norm of a dyadic is defined as

$$\|\mathbf{a}\otimes\mathbf{b}\|^2 = \sum_{i,j=1}^{3}(a_i b_j)^2 . \tag{64}$$

7. Special shapes

Using the results concerning the most general symmetric shape of scatterer, the ellipsoidal one, we can obtain as geometrically degenerate cases shapes like the oblate and prolate spheroids and the sphere. In these cases there are no elliptic integrals and we can obtain simpler formulae. Despite the simpler form of these formulae they are not so simple that they can be used to draw direct conclusions about the behaviour of the scattered field, except for the case of the sphere.

(i) *Spheroids*

For $a_1 \neq a_2 = a_3$ the ellipsoid degenerates to a spheroid, prolate when $a_1 > a_2 = a_3$ and oblate when $a_1 < a_2 = a_3$. In this case the elliptic integrals are elementary and we have

$$I_0^1(\rho) = \frac{1}{h_3}\begin{cases}\frac{1}{2}\ln\left(\frac{\rho+h_3}{\rho-h_3}\right), & a_1 > a_2 ,\\[2mm] \frac{1}{\mathrm{i}}\tan^{-1}\left(\frac{\mathrm{i}h_3}{\rho}\right), & a_1 < a_2 ,\end{cases} \tag{65}$$

$$I_1^1(\rho) = \frac{1}{h_3^2}\left(I_0^1(\rho) - \frac{1}{\rho}\right), \tag{66}$$

$$I_1^2(\rho) = I_1^3(\rho) = -\frac{1}{2h_3^2}\left(I_0^1(\rho) - \frac{\rho}{\rho^2-h_3^2}\right), \tag{67}$$

$$I_2^1(\rho) = \frac{9}{4h_3^4}\left(I_0^1(\rho) - \frac{3\rho}{3\rho^2-h_3^2}\right), \tag{68}$$

$$I_2^2(\rho) = I_2^5(\rho) = \frac{3}{8h_3^4}\left(I_0^1(\rho) - \frac{\rho(3\rho^2-5h_3^2)}{3(\rho^2-h_3^2)^2}\right), \tag{69}$$

$$I_2^3(\rho) = I_2^4(\rho) = -\frac{3}{2h_3^4}\left(I_0^1(\rho) - \frac{3\rho^2 - 2h_3^2}{3\rho(\rho^2 - h_3^2)}\right) \tag{70}$$

where

$$\rho = \begin{cases} h_3 \cosh \omega \\ h_3 \,\mathrm{i} \sinh \omega \end{cases} = \begin{cases} \sqrt{a_1^2 - a_2^2} \cosh \omega \,, & a_1 > a_2 \,, \\ \sqrt{a_2^2 - a_1^2} \sinh \omega \,, & a_1 < a_2 \,, \end{cases} \tag{71}$$

and $(\omega, \theta, \varphi)$ are the spheroidal coordinates related to the cartesian coordinates (x_1, x_2, x_3) by

$$\begin{aligned} x_1 &= \rho \cos \theta \,, & \omega &\in [0, +\infty) \,, \\ x_2 &= \sqrt{\rho^2 - h_3^2} \sin \theta \cos \varphi \,, & \theta &\in [0, \pi] \,, \\ x_3 &= \sqrt{\rho^2 - h_3^2} \sin \theta \sin \varphi \,, & \varphi &\in [0, 2\pi) \,. \end{aligned}$$

For $\rho = a_1$ we obtain

$$I_0^1 = \frac{1}{a_2} \begin{cases} \left[\left(\frac{a_1}{a_2}\right)^2 - 1\right]^{-\frac{1}{2}} \cosh^{-1}\left(\frac{a_1}{a_2}\right), & a_1 > a_2 \,, \\ \left[1 - \left(\frac{a_1}{a_2}\right)^2\right]^{-\frac{1}{2}} \cos^{-1}\left(\frac{a_1}{a_2}\right), & a_1 > a_2 \,, \end{cases} \tag{72}$$

and through (66)–(70) all the other elliptic integrals can be expressed as functions of the ratio a_1/a_2, whenever $\rho = a_1$.

Having the values of he elliptic integrals we can substitute them in the corresponding expressions and obtain the results for an oblate or a prolate spheroid.

The needle-shaped scatterer can be approximated by a prolate spheroid where $a_1 \gg a_2 = a_3$. In this case

$$I_0^1 \sim \frac{1}{a_2} \frac{\ln 2\left(\frac{a_1}{a_2}\right)}{\left(\frac{a_1}{a_2}\right)} \,. \tag{73}$$

In the case where $a_1 \ll a_2 = a_3$, the oblate spheroid takes the shape of a circular disc and

$$I_0^1 \sim \frac{\pi}{2a_2} \,. \tag{74}$$

(ii) *Sphere*

When $a_1 = a_2 = a_3 = a$ the elliptic integrals can be evaluated immediately, and we have

$$I_0^1(\rho) = \frac{1}{\rho} \,, \tag{75}$$

$$I_1^n(\rho) = \frac{1}{3\rho^3}, \quad n = 1, 2, 3, \tag{76}$$

$$I_2^n(\rho) = \frac{1}{5\rho^5}, \quad n = 1, 2, 3, 4, 5. \tag{77}$$

We also have that $\rho = r$, $\mu = \nu = 0$ and $\Lambda = \Lambda' = \alpha^2$. In order to evaluate the undetermined forms in the various expressions it suffices to approximate the sphere by, say, a prolate spheroid setting $\alpha_i = \alpha(1 + \varepsilon)$, $\varepsilon > 0$, $\alpha_2 = \alpha_3 = \alpha$ and obtain the case of a sphere in the limit as $\varepsilon \to 0+$. Using this procedure we arrive after some calculations at the following forms for the scattering cross-section:

(a) *for P-incidence*

$$\begin{aligned}\sigma^p = {} & \frac{\tau_1\kappa_1^4}{12\pi}(\tau_1^3 + 2)\Big(\frac{\rho_2}{\rho_1} - 1\Big)^2 V_2^2 + \frac{\tau_1^4\kappa_1^4}{4\pi}\Big[(1 - 2\tau_1^2)^2\Big(\frac{\lambda_2}{\lambda_1} - 1\Big)^2 \\ & + \frac{4}{3}\tau_1^2(1 - 2\tau_1^2)\Big(\frac{\lambda_2}{\lambda_1} - 1\Big)\Big(\frac{\mu_2}{\mu_1} - 1\Big) + \frac{4}{15}\tau_1^4\Big(\frac{\mu_2}{\mu_1} - 1\Big)^2\Big](2\delta + 3\varepsilon)^2 V_2^2 \\ & + \frac{\tau_1^3\kappa_1^4}{15\pi}[2\tau_1^5 + 5]\Big(\frac{\mu_2}{\mu_1} - 1\Big)^2(4\delta^2 + 3\varepsilon^2 + 4\delta\varepsilon)V_2^2 + 0(\kappa_1^6);\end{aligned} \tag{78}$$

(b) *for S-incidence*

$$\sigma^s = \frac{\tau_1\kappa_1^4}{12\pi}(\tau_1^3 + 2)\Big(\frac{\rho_2}{\rho_1} - 1\Big)^2 V_2^2 + \frac{2\tau_1\kappa_1^4}{15\pi}(2\tau_1^5 + 5)\Big(\frac{\mu_2}{\mu_1} - 1\Big)^2\delta^2 V_2^2 + 0(\kappa_1^6), \tag{79}$$

$$\delta = \frac{3}{2(2\mu_1 + \mu_2)}[\mu_1 + (\mu_1 - \mu_2)\omega], \tag{80}$$

$$\varepsilon = \frac{1}{2(2\mu_1 + \mu_2)}\Big[\lambda_1 - \frac{3\lambda_2(\lambda_1 + 2\mu_1)}{4\mu_1 + 2\mu_2 + 3\lambda_2} - 2(\mu_1 - \mu_2)\omega\Big], \tag{81}$$

$$\omega = \frac{\mu_1(\lambda_1 + 6\mu_1)}{2\mu_1(7\mu_1 + 8\mu_2) + 3\lambda_1(3\mu_1 + 2\mu_2)}. \tag{82}$$

8. Discussion

In this paper a systematic analysis of the low-frequency elastic scattering problem concerning a penetrable ellipsoidal scatterer is presented. Using low-frequency techniques the scattering problem has been transformed to a series of potential problems which can be solved recursively. The first two low-frequency approximations of the displacement field have been derived for the most general closed second-degree geometric figure, the triaxial ellipsoid, in the case where waves are also excited within the scatterer, as well as the leading terms of the normalized spherical scattering amplitudes and of the scattering cross-sections. It turned out that the lack of rotational symmetry for the scatterer renders the problem very difficult to solve in closed analytical form and a new calculational technique had to be introduced.

The physical interpretation of the mathematical problem analyzed in this work involves a plane harmonic elastic wave, longitudinal or transverse, that propagates in the three-dimensional Euclidean space in the presence of a penetrable ellipsoidal scatterer. The existence of such a discontinuity in the elastic properties disturbs the incident wave and as a result waves of both types, longitudinal and transverse, are scattered from the obstacle which, due to the linearity of the problem, are superposed on the existing incident wave. The propagation vector of the incident wave, as well as the polarization vector, is arbitrarily oriented with respect to the principal axes of the ellipsoid. The two problems of longitudinal and transverse incident waves are analyzed simultaneously by introducing a generalized polarization vector.

The real difficulty of the problem lies in the evaluation of the first-order low-frequency approximation, because the zeroth-order approximation could be easily obtained and by intuition. With the proposed technique all the difficulties arising from the complicated equation, the rather involved boundary conditions and the ellipsoidal shape of the scatterer have been overcome. The proposed technique is quite general and can be used for the evaluation of the n-th order approximation of the displacement field, but obviously the difficulty in the calculations increases very rapidly with the order of the approximation field. From the analytical point of view it is clear that the penetrable scattering problem is much harder than the rigid or the cavity problem; this is because there are two boundary conditions that must be satisfied simultaneously and because of the complexity of the surface-stress operator.

The leading-term approximation of the normalized spherical scattering amplitudes is of the order of k^3, as $k \to 0$, and depends on certain surface and volume integrals of $\mathbf{\Phi}_1$. The leading-term approximation of the scattering cross-section is of the fourth order of the wavenumber. The evaluation of the leading term of the scattering cross-section by use of our approach demands the knowledge of the coefficients $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_1$ only and consequently the calculation of the leading term of the scattering cross-section is realistic. If, in order to evaluate the leading-term approximation of the scattering cross-section, the results of [4] have to be used, the evaluation of the coefficients $\mathbf{\Phi}_i$, $i = 0, 1, 2, 3, 4$, in the low-frequency region will be needed. Considering the rapidly increasing difficulty in evaluating the coefficients $\mathbf{\Phi}_n$, our approach is very adequate and efficient.

The case of the cavity can be considered as a special case of a penetrable scatterer and the solution can be obtained by taking $\lambda_2 = \mu_2 = \rho_2 = 0$ and putting the displacement field inside the scatterer equal to zero. Our results are in agreement with existing results concerning the ellipsoidal cavity [11].

From a physical point of view the solution of the described problem has applications to composite materials. A composite material can be considered as a homogeneous isotropic elastic medium containing inclusions. The modeling of these materials is of considerable engineering importance because, by doing so, their mechanical properties can be obtained. So, the solution of the examined scattering problem can be exploited in order to evaluate, using certain energy methods, the elastic moduli of the material. Obviously, the results for the scattering problems with $\mu_1 = \mu_2$, $\nu_1 \neq \nu_2$ and $\mu_1 \neq \mu_2$, $\nu_1 = \nu_2$ (ν is the Poisson ratio), which are of special interest in applications, are special cases of the general problem which we examine.

By the combination of geometrical and physical degeneration, many problems of practical interest can be obtained as special cases of the problem that has been examined. Besides, the results of this paper are expressed in a form suitable for numerical calculations which are actually reduced to numerical evaluation of given functions. On the other hand, many interesting results for special cases can be obtained in closed analytical form.

Finally, we mention that our results can be used further in order to establish lower and upper bounds for the surface-traction integral in the low-frequency region, that is, for the leading-term approximation of the scattering amplitudes for a scatterer of general shape. Such bounds are presented in [16] for the case of a rigid scatterer. Besides, based on our results, especially the far-field pattern (the scattering amplitudes), the inverse scattering problem can be investigated. It is well known that the knowledge of the solutions of the direct problem for arbitrary domains is necessary in order to treat the inverse problem, generally. But if we restrict ourselves to symmetric shapes, we can investigate methods for the solution of the inverse problem based on the far-field pattern behaviour. For the inverse scattering problem in acoustics, using a low-frequency far-field pattern approach, we refer to [17].

Appendix: Ellipsoidal harmonics

The interior ellipsoidal harmonics of degree n are

$$\mathbb{E}_n^m(\rho, \mu, \nu) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \tag{A.1}$$

for $m = 1, 2, \ldots, 2n+1$, where E_n^m are the Lamé functions of the first kind.

The exterior ellipsoidal harmonics of degree n are given by

$$\mathbb{F}_n^m(\rho, \mu, \nu) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \tag{A.2}$$

for $m = 1, 2, \ldots, 2n+1$, where F_n^m are the Lamé functions of the second kind. These are related to $E_n^m(\rho)$ by the formula

$$F_n^m(\rho) = (2n+1)E_n^m(\rho)I_n^m(\rho) \tag{A.3}$$

where

$$I_n^m(\rho) = \int_\rho^{+\infty} \frac{\mathrm{d}u}{[E_n^m(u)]^3\sqrt{u^2 - h_2^2}\sqrt{u^2 - h_3^2}} \tag{A.4}$$

are elliptic integrals.

The Lamé functions E_n^m up to degree two, which are used in this paper, are

$$E_0(u) = 1\,, \tag{A.5}$$

$$E_1^n(u) = |u^2 - a_1^2 + a_n^2|^{\frac{1}{2}}\,, \quad u = \rho, \mu, \nu\,, \quad n = 1, 2, 3\,, \tag{A.6}$$

$$E_2^1(u) = u^2 - a_1^2 + \Lambda\,, \tag{A.7}$$

$$E_2^2(u) = u^2 - a_1^2 + \Lambda' \tag{A.8}$$

where

$$\begin{Bmatrix} \Lambda \\ \Lambda' \end{Bmatrix} = \frac{1}{3}\left\{\sum_{n-1}^{3} a_n^2 \pm \left[\sum_{n=1}^{3}\left(a_n^4 - \frac{a_1^2a_2^2a_3^2}{a_n^2}\right)\right]^{\frac{1}{2}}\right\}, \tag{A.9}$$

$$E_2^{6-n}(u) = \frac{E_1^1(u)E_1^2(u)E_1^3(u)}{E_1^n(u)}, \quad n = 1, 2, 3. \tag{A.10}$$

The cartesian forms of the interior ellipsoidal harmonics up to degree two are

$$\mathbb{E}_1^n(\rho, \mu, \nu) = \frac{h_1 h_2 h_3}{h_n} x_n, \quad n = 1, 2, 3, \tag{A.11}$$

$$\mathbb{E}_2^1(\rho, \mu, \nu) = (\Lambda - a_1^2)(\Lambda - a_2^2)(\Lambda - a_3^2)\left(\sum_{n=1}^{3} \frac{x_n^2}{\Lambda - a_n^2} + 1\right), \tag{A.12}$$

$$\mathbb{E}_2^2(\rho, \mu, \nu) = (\Lambda' - a_1^2)(\Lambda' - a_2^2)(\Lambda' - a_3^2)\left(\sum_{n=1}^{3} \frac{x_n^2}{\Lambda' - a_n^2} + 1\right), \tag{A.13}$$

$$\mathbb{E}_2^{6-n}(\rho, \mu, \nu) = h_1 h_2 h_3 x_1 x_2 x_3 \frac{h_n}{x_n}, \quad n = 1, 2, 3. \tag{A.14}$$

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